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PHYSICS LETTERS B

Physics Letters B 655 (2007) 84–89

www.elsevier.com/locate/physletb

A modularity test for elliptic mirror symmetry

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Received 28 May 2007; received in revised form 3 August 2007; accepted 14 August 2007

Available online 1 September 2007

Editor: M. Cvetič

Abstract

In this Letter a previously initiated program to construct space from modular forms on the string worldsheet is applied to mirror symmetry. Predictions of an algebraic mirror construction are confirmed for elliptic curves of Brieskorn–Pham type by showing that the string theoretic modular forms associated to the Hasse–Weil L-series of mirror pairs of such curves are identical.

PACS: 11.25.-w; 11.25.Hf; 11.25.Mj

MSC: 11G25; 11G40; 14G10; 14G40

Keywords: Varieties over finite fields; L-functions; Zeta functions; Arithmetic varieties; Fundamental strings; Conformal field theory; Compactification

1. Introduction

Mirror symmetry is usually considered in the context of varieties defined over complete fields, such as the complex number field $\mathbb{C}$. It has become clear in the recent past that interesting information about the underlying string physics can be obtained by considering manifolds over fields of different type, in particular those of positive characteristics, because this allows to probe the geometry with tools that are not available over the complex numbers. It turns out, for example, that certain generating functions associated to such arithmetic probes are directly related to generating functions on the string worldsheet, thereby connecting the physics on the string worldsheet theory to arithmetic quantities derived from the geometry of the corresponding varieties. More precisely, it was shown in Refs. [1–4] that it is possible to relate modular forms constructed from the worldsheet conformal field theory to modular forms that arise from the arithmetic of the associated varieties. It was furthermore shown that it is possible to construct Brieskorn–Pham type Calabi–Yau manifolds directly from the modular forms derived from conformal field theories [3,4].

The main motivation for the program developed in the above references has been to gain a better understanding of the emergence of spacetime in string theory. This is an old problem that has previously resisted a concrete formulation amenable to explicit constructions. A second motivation comes from the hope that a more incisive understanding of the relation between the geometry of spacetime and the physics of the worldsheet might also lead to a better understanding of mirror symmetry. It is in the context of exact models that mirror symmetry is represented by very simple operations, and the question becomes whether these operations can be mapped into the geometric framework. The problem of finding an interpretation of mirror symmetry in the context of the congruence zeta function of Artin has been explored in Refs. [5,6], and has been further discussed in [7–10]. This has turned out to be difficult because cycles of different dimensions are encoded in the Artin zeta function in rather different ways. This becomes

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0370-2693
doi:10.1016/j.physletb.2007.08.059

clear by considering the Weil–Grothendieck factorization of the zeta function [11,12]

$$Z(X/\mathbb{F}_p, t) = \frac{\prod_{j=0}^{n} \mathcal{P}_p^{2j+1}(t)}{\prod_{j=0}^{n} \mathcal{P}_p^{2j}(t)}, \tag{1}$$

where $\mathcal{P}_p^j(t)$ are polynomials of degree $\deg \mathcal{P}_p^j(t) = b^j(X)$, where $b^j(X)$ is the jth Betti number of the variety X of complex dimension n. Mirror symmetry implies a map $H^{p,q}(X) \leftrightarrow H^{n-p,q}(\hat{X})$ on the cohomology rings of the mirror pair $(X, \hat{X})$, and therefore $H^n(X) \leftrightarrow \bigoplus_i H^{i,i}(\hat{X})$. On the worldsheet mirror symmetry is a trivial isomorphism, which raises the question whether the construction of spacetime geometry from worldsheet modular forms using the methods considered in Refs. [1–4] can be used to probe mirror symmetry in a different way. The purpose of this note is to apply the physical interpretation of the Hasse–Weil L-function found in these papers to provide such a string theoretic modularity test for the geometric mirror construction described in Refs. [13,14].

The outline of this article is as follows. Section 2 briefly describes the ring isomorphism introduced in [13,14]. In Section 3 this isomorphism is applied to the construction of the algebraic mirrors of the elliptic cubic Fermat curve $E^3 \subset \mathbb{P}_2$. The mirror curves are described by polynomials that are not diagonal, and live in weighted projective spaces. A priori their Hasse–Weil L-functions thus should be expected to be different. The conformal field theory C^3 related to the curve E^3 is, however, isomorphic to the theory corresponding to the mirror curve. The worldsheet modular forms that were found in [1] to provide the building blocks for the elliptic modular form of the Fermat cubic should therefore also enter the geometric modular form of the mirror. Hence the L-function of the algebraic mirror should be identical to that of E^3. This is confirmed in Section 4. A similar analysis can be applied to the remaining elliptic curves of Brieskorn–Pham type, using the results of [2,3].

2. Geometric mirror map

The following paragraphs contain a brief review of the algebraic mirror map introduced in [13,14] to make this article self-contained. The original motivation for this map arose from the observation of cohomological mirror symmetry in the context of weighted projective hypersurfaces described in [16]. In that construction, or rather in the Landau–Ginzburg version of this class, the computation of the Hodge numbers produced a highly symmetric distribution of these numbers, and the question arose whether "Hodge mirror pairs" are actually mirrors. The class of varieties considered in [16] e.g. contains two spaces with Hodge numbers that are mirrors of the Hodge numbers of the quintic. The question then was whether this was accidental or based on an isomorphism of the associated conformal field theory. The strategy of Refs. [13,14] to address this issue was to establish an isomorphism between two differently constructed orbifold rings, which emerge as building blocks in different Calabi–Yau varieties. The original idea for this construction arose from the observation in [15] that the transition from diagonal affine invariants to D-type invariants in the partition function of Gepner models sometimes produces mirror theories. The basic isomorphism of [13,14] is a generalization of this transition to much more general Landau–Ginzburg theories.

The first ring is defined via the quotient

$$\mathcal{R} = \mathbb{C}_{(\frac{b}{g_{ab}}, \frac{a}{g_{ab}})}[x_1, x_2]/\mathcal{I}_{a,b}, \tag{2}$$

where the ideal $\mathcal{I}_{a,b}$ is defined by the polynomial

$$p_{a,b}(x_1, x_2) = x_1^a + x_2^b. \tag{3}$$

Here g_{ab} is the greatest common divisor of a and b. The orbifold ring $\mathcal{O}$ derived from $\mathcal{R}$ is then defined via the cyclic group $G_b = \mathbb{Z}/b\mathbb{Z}$ as

$$\mathcal{O} = \mathcal{R}/G_b, \tag{4}$$

where the action of G_b is defined as

$$A: \quad (x_1, x_2) \mapsto \left(\xi^{b-1} x_1, \xi x_2\right). \tag{5}$$

The second ring is constructed via

$$\hat{\mathcal{R}} = \mathbb{C}_{(\frac{b^2}{h_{ab}}, \frac{a(b-1)-b}{h_{ab}})}[y_1, y_2]/\hat{\mathcal{I}}_{ab}, \tag{6}$$

where the ideal $\hat{\mathcal{I}}_{a,b}$ is defined by the polynomial

$$\hat{p}_{a,b}(y_1, y_2) = y_1^{a(b-1)/b} + y_1 y_2^b, \tag{7}$$

and h_{ab} denotes the greatest common divisor of b^2 and $(ab - a - b)$. The orbifold ring $\hat{\mathcal{O}}$ derived from $\hat{\mathcal{R}}$ is defined as

$$\hat{\mathcal{O}} = \hat{\mathcal{R}}/G_{b-1}, \tag{8}$$

where the action of G_{b-1} is defined as

$$\hat{A}:\quad (y_1, y_2) \mapsto \big(\xi y_1, \xi^{b-2} y_2\big). \tag{9}$$

One can show that there exists a 1–1 transformation that maps these orbifold rings into each other [14], leading to the following result.

Proposition. *The rings $\mathcal{O}$ and $\hat{\mathcal{O}}$ are isomorphic.*

More explicitly, the basic isomorphism can be summarized as

$$\begin{aligned}
&\mathbb{C}_{(\frac{b}{g_{ab}}, \frac{a}{g_{ab}})}\left[\frac{ab}{g_{ab}}\right] \ni \big\{z_1^a + z_2^b = 0\big\}/G_b : \big[(b-1)1\big] \\
&\quad\cong \mathbb{C}_{(\frac{b^2}{h_{ab}}, \frac{a(b-1)-b}{h_{ab}})}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \big\{y_1^{a(b-1)/b} + y_1 y_2^b = 0\big\}/G_{b-1} : \big[1(b-2)\big],
\end{aligned} \tag{10}$$

where $G_b : [(b-1)1]$ denotes the group element of G_b acting as defined in (5).

This basic isomorphism can be applied to algebraic varieties of any dimension, where it can lead to a number of different phenomena, depending on how the quotient construction involved combines with the (weighted) projective invariance of the ambient space. It can happen, in particular, that the symmetries of the affine surface defining the quotients of the image theory become part of the weighted projective equivalence when the singularities just described are embedded in Calabi–Yau varieties. The resulting spaces can then become mirrors of each other if the resolution of the singularities produces the appropriate cohomological structure.

The simplest application of the strategy just outlined is provided by 3-folds for which the basic isomorphism itself gives the mirror map, without the necessity of an iterative application. Such an example is provided by the mirror configuration

$$\mathbb{P}_{(3,8,33,66,88,132)}[264]^{(57,81)}/G_2 \sim \mathbb{P}_{(3,8,66,88,99)}[264]^{(81,57)}, \tag{11}$$

discussed in [15].

In general an iterative application of the basic map is necessary to construct the mirror manifold. This can be illustrated by constructing the algebraic mirror of the quintic threefold family

$$X_\lambda = \left\{(z_0 : \cdots : z_4) \in \mathbb{P}_4 \,\middle|\, \sum_i z_i^5 + \lambda \prod_i z_i = 0\right\}, \tag{12}$$

by applying the ring isomorphisms iteratively as follows. The quotient construction of the exact model mirror [17] suggests to consider the quotient by the product of four cyclic groups G_5 via the following action

$$G_5^4:\quad \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix} \tag{13}$$

where the notation [0 0 4 1 0] e.g. means the action

$$(z_0, z_1, z_2, z_3, z_4) \mapsto \big(z_0, z_1, \xi^4 z_3, \xi z_3, z_4\big), \tag{14}$$

where $\xi \in \mu_5$ is a fifth root of unity. The ring isomorphisms (10) then lead to the mirror manifold

$$X'_\lambda = \left\{(y_0 : \cdots : y_4) \in \mathbb{P}_{(64,48,52,51,41)} \mid y_0^4 + y_0 y_1^4 + y_1 y_2^4 + y_2 y_3^4 + y_3 y_4^5 + \lambda \prod_i y_i = 0\right\}. \tag{15}$$

These examples generalize to many families in the class of weighted projective space constructed in [16,18,19].

3. Mirror families of deformed elliptic Brieskorn–Pham curves

Denote by $A_{1,k}^{(1)}$ the affine Lie algebra at conformal level k based on $\mathrm{sl}(2,\mathbb{C})$, and let C^d for $d = 3, 4, 6$ be the three exactly solvable conformal field theory models at central charge $c = 3$. They are given by the GSO projected tensor products of $N = 2$ minimal superconformal models based on $A_{1,k}^{(1)}$ [20]

$$C^3 = \big(A_{1,1}^{(1)}\big)^{\otimes 3}_{\mathrm{GSO}}, \qquad C^4 = \big(A_{1,2}^{(1)}\big)^{\otimes 2}_{\mathrm{GSO}}, \qquad C^6 = \big(A_{1,1}^{(1)} \otimes A_{1,4}^{(1)}\big)_{\mathrm{GSO}}. \tag{16}$$

The notation of these models is motivated by the Landau–Ginzburg analyses of Refs. [21–23], according to which these models are expected to be related to the elliptic Brieskorn–Pham curves

$$\begin{aligned} E^3 &= \{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0\}, \\ E^4 &= \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^4 + z_1^4 + z_2^2 = 0\}, \\ E^6 &= \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,2,3)} \mid z_0^6 + z_1^3 + z_2^2 = 0\}. \end{aligned} \tag{17}$$

It was shown in [3] that the curves E^d can be constructed directly from the conformal field theories C^d via the modular forms that enter their partition functions.

Applying the isomorphism (10) to the cubic family

$$E_\psi^3 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 - 3\psi z_0 z_1 z_2 = 0\} \tag{18}$$

leads to the elliptic mirror family $\hat{E}_\psi^3 = E_\psi^3 / G_3$, where the action of the group $G_3 \cong \mu_3$ is given by

$$G_3: \quad (z_0, z_1, z_2) \mapsto (\xi_3^2 z_0, \xi_3 z_1, z_2). \tag{19}$$

The group action of the image theory under the basic isomorphism becomes part of the weighted projective equivalence, hence the algebraic image of this mirror is given by

$$\hat{E}_\psi^3 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(3,1,2)} \mid z_0^2 + z_0 z_1^3 + z_2^3 - 3\psi z_0 z_1 z_2 = 0\}. \tag{20}$$

There exists a second quotient, which can be constructed by considering two μ_3 groups, with an action given by

$$G_3^2: \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}. \tag{21}$$

Applying the basic ring isomorphism iteratively leads to the algebraic form of the mirror given by

$$\hat{E}_\psi^{3'} = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(2,1,1)} \mid z_0^2 + z_0 z_1^2 + z_1 z_2^3 - 3\psi z_0 z_1 z_2 = 0\}. \tag{22}$$

The results of [3] show that the L-functions of the Brieskorn–Pham curves $E^6 \subset \mathbb{P}_{(1,2,3)}$ and $E^4 \subset \mathbb{P}_{(1,1,2)}$ are different from the L-function of E^3. Because $\hat{E}^3$ and $\hat{E}^{3'}$ are algebraic mirrors of the Fermat cubic E^3, their L-functions should agree with the L-function of E^3, even though they are hypersurfaces in the same weighted projective planes as E^6 and E^4, respectively.

Similar considerations apply to the algebraic image of the quotient of the quartic family E_ψ^4 and the degree six family E_ψ^6. For E_ψ^4 the algebraic mirror map leads to the family

$$\hat{E}_\psi^4 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(2,1,3)} \mid z_0^3 + z_0 z_1^4 + z_2^2 - 4\psi z_0 z_1 z_2 = 0\}, \tag{23}$$

while the algebraic mirror of E_ψ^6 is

$$\hat{E}_\psi^6 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^4 + z_0 z_1^3 + z_2^2 - 6\psi z_0 z_1 z_2 = 0\}. \tag{24}$$

In both cases the symmetry actions on the image theories are trivial again.

For these examples mirror symmetry again predicts that the L-functions of $\hat{E}^4 \subset \mathbb{P}_{(1,2,3)}$ and $\hat{E}^6 \subset \mathbb{P}_{(1,1,2)}$ should agree with the L-functions of $E^4 \subset \mathbb{P}_{(1,1,2)}$ and $E^6 \subset \mathbb{P}_{(1,2,3)}$ respectively, even though the L-functions of the Brieskorn–Pham points in these weighted ambient spaces are different. The expectations from the algebraic mirror map are confirmed in the next section.

4. The Hasse–Weil L-function of elliptic mirror pairs

A detailed review to the Hasse–Weil L-function can be found in many references, e.g. [24] (a summary with focus on elliptic curves can be found in [3]). Briefly, the Hasse–Weil L-function of an algebraic curve X is determined by the local congruence zeta functions at all prime numbers p. This is defined in [11] as the generating series

$$Z(X/\mathbb{F}_p, t) = \exp\left(\sum_{r \in \mathbb{N}} N_{r,p}(X) \frac{t^r}{r}\right), \tag{25}$$

where $N_{r,p}(X) = \#(X/\mathbb{F}_{p^r})$ denotes the cardinality of the variety over the finite extension $\mathbb{F}_{p^r}$ of the finite field $\mathbb{F}_p$ of characteristic p for any rational prime p, and t is a formal variable. The congruence zeta function admits a cohomological interpretation (1), first envisioned by Weil [11], and later shown by Grothendieck [12] via his theory of étale cohomology. In the present context it is a classic result by F.K. Schmidt that $Z(X/\mathbb{F}_p, t)$ is a rational function, taking the simple form

$$Z(X/\mathbb{F}_p, t) = \frac{\mathcal{P}_p(t)}{(1-t)(1-pt)}, \tag{26}$$

Table 1
The coefficients $\beta_1(p) = N_{1,p}(\hat{E}^3) - (p+1)$ of the elliptic cubic curve $\hat{E}^3$ in terms of the cardinalities $N_{1,p}$ for the lower rational primes

Prime p	2	3	5	7	11	13	17	19	23	29	31
$N_{1,p}$	3	4	6	9	12	9	18	27	24	30	36
$\beta_1(p)$	0	0	0	1	0	−5	0	7	0	0	4

where $\mathcal{P}_p(t)$ is a quadratic polynomial.

More relevant from a physical perspective than the local zeta functions is the global zeta function, obtained by setting $t = p^{-s}$ and taking the product over all rational primes at which the variety has good reduction. Denote by S the set of rational primes at which X becomes singular and denote by P_S the set of primes that are not in S. The global zeta function can be defined as

$$Z(X,s) = \prod_{p \in P_S} \frac{\mathcal{P}_p(p^{-s})}{(1-p^{-s})(1-p^{1-s})} = \frac{\zeta(s)\zeta(s-1)}{L(X,s)}, \tag{27}$$

where the Hasse–Weil L-function has been introduced as

$$L(X,s) \doteq \prod_{p \in P_S} \frac{1}{\mathcal{P}_p(p^{-s})}, \tag{28}$$

and $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ is the Riemann zeta function of the rational field $\mathbb{Q}$. Here $\doteq$ denotes the L-function up to a finite number of primes. The treatment of the finite number of exceptional primes is more elaborate and can be found e.g. in [3], leading to the completion of the L-functions of the curves E^d at those primes for which they are singular.

In Refs. [1–3] the Hasse–Weil L-functions $L_{\mathrm{HW}}(E^d,s)$ of the curves E^d, $d = 3,4,6$ were computed explicitly, leading to three distinct series. It was shown that all three corresponding modular cusp forms of weight two factor into string theoretic forms twisted by characters that are associated to number fields that are determined by the quantum dimensions of the conformal field theory. More precisely, denote by $S_2(\Gamma_0(N))$ the space of cusp forms of modular level N and weight two with respect to the congruence group $\Gamma_0(N) \subset \mathrm{SL}(2,\mathbb{Z})$. Then the modular forms associated to the L-series of E^d are elements $f(E^d,q) \in S_2(\Gamma_0(N_d))$ with $N_d = 27, 64, 144$ for $d = 3,4,6$ respectively. Their string theoretic factorizations take the form

$$f(E^3,q) = \Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9), \qquad f(E^4,q) = \Theta^2_{1,1}(q^4)^2 \otimes \chi_2, \qquad f(E^6,q) = \Theta^1_{1,1}(q^6)^2 \otimes \chi_3, \tag{29}$$

where

$$\Theta^k_{\ell,m}(\tau) = \eta^3(\tau) c^k_{\ell,m}(\tau) \tag{30}$$

are Hecke indefinite modular forms constructed from Kac–Peterson string functions $c^k_{\ell,m}(\tau)$ associated to the affine Lie algebra $A_1^{(1)}$ [25]. This shows that the arithmetic method is precise enough to detect the different details of the underlying conformal field theory, even in the elliptic framework.

The factors $\mathcal{P}_p(t)$ can be obtained by expanding Weil's defining form of the congruence zeta function and comparing the coefficients to the expansion of Schmidt's rational form of it. Writing the polynomials $\mathcal{P}_p(t)$ at the good primes as

$$\mathcal{P}_p(t) = 1 + \beta_1(p)t + pt^2, \tag{31}$$

the coefficient $\beta_1(p)$ is expressed in terms of the cardinalities $N_{1,p} = \#(X/\mathbb{F}_p)$ as

$$\beta_1(p) = N_{1,p} - (p+1). \tag{32}$$

The elliptic modularity theory proven in [26,27] guarantees that the computation of a finite number of cardinalities is sufficient to determine the corresponding modular forms completely. For the elliptic mirror curve $\hat{E}^3$ the results for the numbers $N_{1,p}$ for the first few primes are collected in Table 1.

Inserting these cardinalities into the Hasse–Weil series of the mirror curve $\hat{E}^3$ of the cubic Fermat curve leads to

$$L_{\mathrm{HW}}(\hat{E}^3,s) = 1 - \frac{2}{4^s} - \frac{1}{7^s} + \frac{5}{13^s} + \frac{4}{16^s} - \frac{7}{19^s} + \cdots. \tag{33}$$

Comparing this result with the Hasse–Weil L-function computed in [1,3] shows agreement. It can similarly be shown that the L-function of the curve $\hat{E}^{3'}$ agrees with that of E^3.

The mirrors $\hat{E}^4$ and $\hat{E}^6$ furthermore have the same L-function as E^4 and E^6, respectively. Summarizing, the curves E^d have the property that $L_{\mathrm{HW}}(\hat{E}^d,s) = L_{\mathrm{HW}}(E^d,s)$. Expressed in terms of the associated modular forms therefore leads to the following result.

Proposition. *The modular forms associated to mirror pairs* $(E^d, \hat{E}^d)$ *of elliptic Brieskorn–Pham curves* E^d *of degree* $d = 3, 4, 6$ *satisfy the identity*

$$f(\hat{E}^d, q) = f(E^d, q). \tag{34}$$

5. Concluding remarks

It follows from the proposition that for elliptic curves the arithmetic structure of the underlying geometry correctly captures the worldsheet theoretic nature of mirror symmetry, contrary to appearances in the cohomological form of the zeta function. It was shown in [3] that the three elliptic Brieskorn–Pham curves considered here can be constructed from the string theoretic modular forms of the exactly solvable conformal field theory on the worldsheet. The result shown thus establishes that for elliptic curves the construction of the internal space from string modular forms along the lines of [1–4] extends to mirror pairs of such manifolds. The generalization to higher dimensional varieties is complicated by the fact that the Hasse–Weil L-function is not the relevant object to consider because there are in general several irreducible motives, each leading to an L-function. The strategy in that case should be to find modular motives, along the lines described in [4], for both varieties in a mirror pair, and to trace the construction of these motives to the modular forms on the worldsheet, if the resulting modular forms admit a string theoretic interpretation.

Acknowledgements

It is a pleasure to thank Monika Lynker for conversations and Jack Morse for correspondence. I have benefitted from a Scholar Grant at the Kavli Institute for Theoretical Physics and I am grateful to the KITP for support. Part of this work was funded by a KSU Incentive Grant for Scholarship, an IUSB Faculty Research Grant, and the National Science Foundation under Grant No. PHY05-51164.

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